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Celestial Mechanics: Never Say No To A Computer

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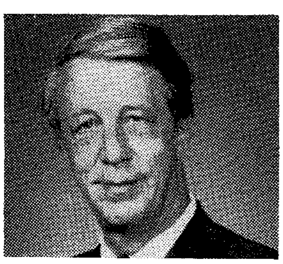
BY the middle of the twentieth century, celestial mechanics had retired from academic life. For three hundred years, it had been a source of inspiration for mathematicians, and a leader in numerical analysis. But Poincaré's *Méthodes Nouvelles* brought mathematical astronomers to the threshold of inextricable problems while, at the same time, celestial mechanics encountered its limits in dealing with observers. Delaunay had spent twenty-five years of his life attempting to complete a solution to the main problem in lunar theory, and no sooner was it in print than it proved to be well below the standards of precision for optical observations. Nevertheless, the challenge was picked up again by Brown who thought Hill had discovered a shortcut. He spent fifteen years in solving the main problem, and ten more years adding to it the effects due to the planets, and still seven more years rearranging his formulas so that clerks in the almanac offices could calculate the position of the moon every four hours in less time than it took the moon to move to the next predicted location. Eventually, in 1925, the International Astronomical Union adopted Brown's *Tables of the Moon* as the basis of the lunar ephemerides. Full of pride and expectation, after 32 years of uninterrupted labor, Brown set his students at Yale, Dirk Brouwer and Wallace Eckert, to the task of reducing the observations against his theory. History repeated itself: as they did for Hansen and Delaunay, observations denied to Brown's lunar theory its claim to becoming the primary standard of time.

By then, celestial mechanics had lost its academic standing in most departments of astronomy. It survived in a few fiefdoms: Brouwer at Yale, Herrick at the University of California at Los Angeles, Clemence at the U.S. Naval Observatory, Jefferies in Cambridge, Chazy in the Sorbonne, Stracke in Berlin, Subbotin at the Institute of Theoretical Astronomy of the U.S.S.R. Academy of Sciences in Leningrad, Elie Stromgren in Copenhagen, Milankovitch in Belgrade, Hagihara in Tokyo, and, *primus inter pares,* Wallace Eckert of the IBM Thomas Watson Research Center at Columbia University.

The University of Louvain in Belgium was one of these few unlikely places where research in celestial mechanics was still active in the mid 1950's. There reigned Monseigneur Georges Lemaître, President of the Pontifical Academy of Sciences in the Vatican. He had made his fame in astronomy when he explained Hubble's law for the red-shift in galaxies as an effect of the expansion of the universe.[1] He had gone a step further in proposing[2-8] that the singularity in Einstein's equations be interpreted as the Big Bang of a primeval atom. Cosmic rays, in his opinion,[9] were the remnants of the initial fireball. In support of his conjecture, he began exploring numerically the orbits of charged particles in the field of a magnetic dipole. The purpose was to explain how particles coming from far away approach the singularity at the dipole. Central to the computing project was Poincaré's view that families of periodic orbits are the armature of the body of tori of conditionally periodic orbits that is the phase space of a Hamiltonian system. Although he had been a student of Charles de la Vallée-Poussin, Lemaître had little time for abstract mathematics; there was nothing he enjoyed more than "manipulating" mathematics. His computing efforts followed very closely the progress in the computing technology. He helped Vanevar Bush at M.I.T. by testing the Differential Analyzer on the Stoermer problem.[10] He moved his calculations of orbits from tables of logarithms to hand-cranked adders, then to electric desk-machines and mechanically automated accounting machines,[11] and finally realized his dream when Hartree gave him access to the vacuum tube computer in development at Cambridge University. What a spectacle it was: a man who began his career by reading Poincaré's *Methodes Nouvelles* in Flanders fields while he served as a cyclist in a communication unit of the Belgian Army during World War I, now a champion of the Big Bang Theory against the Cosmology of the Continuous Creation—declining to program his problems by proxy through graduate students, but leading us in the conviction that the future in celestial mechanics belonged to a new breed of mathematical astronomers: men with a solid background in pure analysis, artists in computer programming, experts in numerical analysis, always on the alert to discover how progress in hardware technology would help in taking up the next challenge.

There was indeed a big challenge on the horizon: Delaunay's analytical solution of the main problem of lunar theory, i.e. 800 pages *in quarto* of formulas in two volumes of the *Mémoires de l'Académie Impériale des Sciences de Paris.*



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I entered the project very cautiously. In the restricted problem of three bodies (a minor planet in the gravity field of the Sun and Jupiter), at the equilateral equilibrium L4, one observes a cluster of rocks called the Trojan planets. They are trapped there. According to Poincaré, the island of stability should be supported by a network of families of periodic orbits. To that structure belong the families which, by virtue of Liapunov's theorem, emanate from the equilibrium itself. Together with my graduate students at Louvain, Jacques Henrard, Jacques Roels, and the late André Delie, I developed programs in FORTRAN II to run on an IBM 1620, and we expanded the Liapunov families as Fourier series whose coefficients were produced as unevaluated polynomials in the parameter of the family.[12-16] To my knowledge, this was the first instance of a literal development by machine that had more substance than an assignment to a graduate class in celestial mechanics. Around these results were developed tools for integrating variational equations,[17] for calculating characteristic exponents of periodic orbits,[18] and for regularizing binary collisions.

For his Ph.D. dissertation, Roger Broucke undertook to revise and to extend[19] the collection of periodic orbits that Elie Stromgren had amassed at the Copenhagen Observatory between 1920 and 1940. The computing power available in Belgium in the late 1950's was very limited. But Broucke had found a way for running his algorithms as a background job on the Bull Gamma-Tambour computer of his employer in Brussels (an arc of periodic orbit whenever the machine was waiting for its next accounting report!). I must say, the work we did in Louvain under the benign supervision of Monseigneur Lemaître, who was then dying of leukemia, vindicated his philosophy: a mathematical directive implemented analytically by computer should control a numerical exploration aimed at extracting precise results to be returned to mathematicians as the impugnable basis for significant conjectures.

A case in point is Brown's genealogy of the families of periodic orbits for Trojan planets. According to Brown, the Liapunov family of long periodic orbits emanating from the equilibrium would terminate with a homoclinic orbit doubly asymptotic to the collinear equilibrium L3. Then it would become the natural family of horseshoe-shaped orbits ending with a homoclinic orbit doubly asymptotic to the collinear equilibrium L2. A bifurcation would occur at this point, with one branch made of orbits enclosing both points L3 and L2, and the other branch made of hourglass orbits. The latter would eventually encounter a homoclinic orbit doubly asymptotic to the collinear equilibrium L1. Another branching, according to Brown, occurs at this point; it leads to periodic planetary orbits around the sun and to periodic satellite orbits around Jupiter. All technicalities set aside, let it be said that Brown's genealogy is wild, unfounded, but widely accepted—in short, a typical instance of Ockham's razor in Bittremieux's version: "Dans les choses où il est impossible de rien savoir, les arguments probables finissent toujours par engendrer la certitude." ("In matters where it is impossible to know anything, probable arguments end up generating certainties.")

At Yale University, in the Christmas recess of 1965, Julian Palmore and I designed an algorithm to continue numerically natural families of nonsymmetric orbits past the point where their analytical representation in d'Alembert series ceases to be valid. We applied it to search for the natural end to the family of short period orbits emanating from the equilateral equilibrium. Fortunately, in that season, the direct coupling IBM 7040-7094 at the Watson Computing Center was "leaking bits" from its memory, and all heavy users had stepped out waiting for the trouble to be fixed. Rigging our programs with all kinds of internal safeguards, we were able to discriminate genuine results from nonsensical runs. The turnaround was so fast for us that, practically, we spent our Christmas vacation in a cubicle next to the ailing machine. So much material was amassed in so little time that we perceived immediately that the short-period Trojan orbits invalidated Brown's conjecture; we even gambled that the family ended with a periodic orbit emanating from the collinear equilibrium. We checked our conjecture and thus produced the first instance[20] of a symmetric orbit branching into two families of nonsymmetric orbits. The following summer, Julian Palmore came to the Boeing Scientific Research Laboratories in Seattle to widen the credibility gap we had opened in Brown's theory, and I went on feeding numerical conjectures to Kenneth Meyer and Dieter Schmidt[21] at the University of Minnesota for them to deduce mathematical propositions about bifurcations of periodic points. The qualitative riddles suggested by Poincaré were beginning to yield to the pressure of the combined analytical and numerical information obtained by computer.

Meanwhile our ability for executing analytical developments by computer was growing. Algebraic manipulations in an automated way may be approached from two directions: either as a problem of matching and substituting strings of characters through an applicative language (FORMAC was the leading contender at the time), or as a problem of applying algebraic laws of operation on fixed data structures. Because I was concerned with solving the well typified equations of celestial mechanics, I opted for the algebraic approach. With Arnold Rom and J.M.A. Danby, I developed a package of FORTRAN subroutines to automatically manipulate multiple Fourier series whose coefficients are multivariate Laurent series. These objects I called *Poisson series,*[22] not because of a historical connection with the work of the famous French mathematician, but to give them a name that an English mouth cannot pronounce, as a retribution for the smile my French accent draws from my very Oxonian coauthor Tony Danby. Poisson series are now a household word in mechanics. The original package, dubbed MAO for Mechanized Algebraic Operations, has since undergone many avatars: Rom translated[23] it into assembler language for the IBM 360-44, then extended[24] it to handle echeloned series. We lent it to Broucke[25] at the Jet Propulsion Laboratory, Van Flandern at the U.S. Naval Observatory, and others[26] who derived from it their personal version. Lately, in Cincinnati, I redesigned the package from scratch and coded it in PL/I. The new concepts in rejuvenated MAO have served Dieter Schmidt in assembling an extraordinarily efficient program to manipulate multivariate polynomials[27] over the field of complex numbers, and Robert Dasenbrock at the U.S. Naval Research Laboratory in adapting his own processor to echeloned series. Nowadays, processors of Poisson series belong to the toolbox of celestial mechanics.

Each step in the development of MAO marked a progress in the automation of asymptotic expansions in celestial mechanics and nonlinear dynamics. A breakthrough occurred during my visit as resident scientist to the Bell Telephone Laboratories in Whippany in the summer of 1968.

From an algorithm invented by Lindstedt for generating periodic solutions to nonlinear differential equations, Poincaré had derived a procedure for building canonical transformations to normalize Hamiltonian systems. Brouwer who learned about this "méthode nouvelle" in von Zeipel's theory of minor planets, applied it skillfully to the theory of artificial satellites, and made it popular among astronomers under the name of von Zeipel.

Automating the procedure revealed its weak point: it requires inverting half the transformation equations and substituting the result in the other half of the equations in order to display the normalizing mapping in its explicit form. The way Jacques Henrard in his Ph.D. dissertation built normalizations in an explicit manner by the method of undetermined coefficients showed me how to recondition Poincaré's method in order to obtain the canonical transformation immediately in its explicit form. The most interesting feature of a perturbation algorithm based on a Lie transformation[28-30] is that the operations may be carried out most naturally in a recursive manner, which makes for a simple transposition into a computer program. There is a natural correspondence[31] between Lie transformations and the canonical transformations built through Poincaré's perturbation method. I learned recently that the concept of Lie transformation has made its way into semiclassical quantum mechanics, and also in the infinitesimal theory of Lie groups where it might be equivalent to the classical Baker-Campbell-Hausdorff[32] formula for composing exponential mappings.

Upon my return to Seattle from the Bell Telephone Laboratories, I addressed myself to the task of restructuring my computer programs around Lie transformations. With Jacques Henrard and Arnold Rom, we rebuilt our procedure for Birkhoff's normalization. In the case of the Trojan planets, we[33] pushed it to order 26, and this gave us at last the capability of deciphering the structure of the phase space around the equilateral equilibrium. Brown's conjecture was totally destroyed; Stromgren's program of classifying the major families of periodic orbits in the restricted problem of three bodies was completed in the case of the sun and Jupiter. Eventually we succeeded in establishing the kind of bifurcations the phase space undergoes as the mass ratio varies.[34]

Only ten years after the launching of Sputnik I, celestial mechanics was alive and well, fully reawakened from its lethargy through the first half of the century. Whereas Derral Mulholland of the University of Texas at Austin worked on setting up a Division of Dynamical Astronomy within the American Astronomical Society to group those members interested in celestial mechanics and astrometry, I conceived the project of founding a scientific journal dedicated to mathematical astronomy. Rallying colleagues around that undertaking has not been an easy task, although Mr. Reidel in Dordrecht, Professor Kopal at the University of Manchester, and Professor Schwarzchild of Princeton University, then president of the American Astronomical Society, immediately gave their wholehearted support. *Celestial Mechanics* was launched in September of 1968 at the semiannual conference of astrodynamics held by the Special Systems Branch in the Laboratory for Theoretical Studies at NASA Goddard Space Flight Center. Dr. Peter Musen must be credited with having given the editorial board of the journal a truly democratic character.

At about the same time, my team in Seattle felt confident it could attack the lunar theory. David Barton in Cambridge University was already busy reproducing some of the 50 operations by which Delaunay eliminated stepwise the periodic arguments in the perturbation. On our side, we felt that we should do more than just reproduce Delaunay's hand calculations of a hundred years ago. We set our goals on producing a solution precise to 50 cm in distance, quite a jump from the 300 km of error in Delaunay's results and the 3 km inaccuracy left in Brown's theory. As we should have no way of checking our results until we reached the final stage, we made sure once more that our basic software tools were adequate, this time by solving[35] the main problem of artificial satellite to the fourth order. As we were definitely ready, the Boeing Company entered a severe financial crisis. That we were able to complete the project at all we owe first and foremost to the quiet intrepidity and the personal integrity of Dr. Burton Colvin, the head of the Mathematics Laboratories.

Our calculations, completed in 18 months, vindicated Delaunay; in his reduced Hamiltonian, we found only one mistake,[36,37] due to an incorrect reduction of fractions at the very last stage. Not only did we produce the mean motions[38] and the complete analytical expressions for the longitude, latitude, and parallax of the moon,[39] but we attached to them the partial derivatives with respect to the constants of the theory. In the long history of the lunar theory, it had never been done before.

For this research, the U.S. National Academy of Sciences awarded me the James Craig Watson Golden Medal.

Yet, by its complexity and its length, the solution created for us more problems than it had resolved. Could it be that a semianalytical solution in the manner of Hill-Brown could be less bulky, hence better suited for inclusion of the planetary perturbations? Judging from the Ph.D. dissertation of Daniel Standaert,[40] which inserts direct planetary perturbations on top of the semianalytical solutions recently developed by Jacques Henrard[41] at the University of Namur and the Chapronts at the Bureau des Longitudes in Paris, one may be tempted to answer the question in the affirmative. Also the brilliant expansion of Brown's theory by Dieter Schmidt[42] at the University of Cincinnati would tend to confirm this position.

For my part, I would like to disagree with the advocates of semi-analytical theories and the schools of numerical averaging. After I left the Boeing Scientific Research Laboratories, I resolved to address the issue in a top-down structured approach. First I considered how, from an analytical solution, one derives ephemerides to be evaluated in real time, possibly in microprocessors on board a satellite or on site at the foot of an antenna. I showed that it can be done for all the tables in the *Nautical Almanac and American Ephemeris* by compressing the multivariate Fourier series into sequences of Chebychev polynomials in a normalized time.[43] The technique has been adopted first by the Bureau des Longitudes in Paris to rejuvenate the *Connaissance des Temps* and modernize the navigation calculations for the French Navy and Air Force. Then it caught favor at the U.S. Naval Observatory for what is called the *Almanac for Computers*; the Institute of Theoretical Astronomy in Leningrad has just announced that it will henceforth publish the *Astronomical Year-Book of the U.S.S.R.* by tables of Chebychev polynomials covering wide intervals of time.[44]

The next question upstream from the terminal product at the user's level was that of evaluating multivariate Fourier series to produce the pivotal points requested by a best approximation in the sense of Chebychev for a compression by orthogonal polynomials. With the collaboration of Shannon Coffey, at the time completing his Ph.D. dissertation in Analysis at the University of Cincinnati, and under the ingenuous prodding of Kenneth Meyer, "helmsman" of our department, I devised a simple technique whereby a FORTRAN program processes the file of an analytical theory to generate automatically a FORTRAN program in an almost optimum manner—not only in CPU time but also in memory allocations.[45] The conventional technique of deriving a set of computing directives (called *Tables*) from an analytical solution (called *Theory*) is now to be replaced by a computer program which converts automatically a list of terms into an almost optimal set of evaluation procedures.

I am now back at the central issue: is there a way of cutting short the traditional developments of celestial mechanics? Of course the question is too wide to deserve an answer. Asterix once observed in typical Breton facetiousness that Obelix learned lifting menhirs by picking up pebbles from the village brook. (Dear Reader, what Asterix meant to say is that big oaktrees grow from but little acorns.)

Trying to understand Aksnes' solution to the main problem in the theory of artificial satellites, I missed a detail at a certain point in the deduction, which sent me off track to an answer at odds with Aksnes' result. Since I had to return from the Mathematics Research Center at the University of Wisconsin-Madison back to my classes at the University of Cincinnati, I concluded that I had made a mistake. Six months later, to find a diversion to the all-absorbing task of transferring the Goddard Trajectory Determination System from the Department of Aerospace Engineering at the University of Texas in Austin to the Time and Frequency Division of the National Bureau of Standards in Boulder, I took another look at my notes on Aksnes' intermediary. What

I had regarded as an error showed itself for what it was: a new canonical transformation to convert perturbations in $1/r^3$ into perturbations in $1/r^2$. This is the *elimination of the parallax*[46]: executed by computer, it proved to be a dramatic short-cut which abbreviated the conventional expansions of Brouwer and Kozai by 87%.[47] From thereon, Terry Alfriend and Shannon Coffey have extended the transformation to the perturbations induced by the zonal coefficients to the eighth order, and they now attack the awesome problem of normalizing the sectoral perturbations.[48]

The neocomputational period in celestial mechanics will consist primarily in the amplification of the formal analytical software, the adaptation of these systems to simplify and accelerate the solving of classical problems and the consideration of new quantitative questions that cannot presently be asked.

The software currently available is inadequate. On the one hand, MAO and its successors operate exclusively over the real or complex algebra of Poisson series—a very wide set of algebraic expressions, though, since it encompasses polynomials, truncated power and Laurent series, as well as Fourier series. Extending the domain to, say, elliptic functions and integrals might require a major overhaul of the software approach; very likely it will determine giving up procedural or algorithmic languages for applicative or functional languages.[49,50]

On the other hand, significant results have been produced thus far by codes operating in the batch mode. Nonetheless they do not offer the mathematician the opportunity of examining intermediary results while they are being produced, of directing simplifications at the terminal while the program executes, or of modifying procedures in the course of execution. For him, the computer is made to simulate a bureau of office clerks: their sole role is to execute programs assigned to them and to report only when the task has come to completion. In fact, we need interactive systems to make *heuristic* programming an integral part of mathematical exploration in nonlinear mechanics. Only at that condition will we be able to produce automated algorithms endowed with the property that they are outlines of subsequent theorem-proving methods rather than the interpretation of antecedent mathematical procedures.

A case in point is the study of semi-simple resonances of type (1-1). A kind of dialogue at the terminal between MAO and the problem of a Keplerian system in a uniform field led, almost by accident, to the discovery of a new set of canonical elements through what I refer to as a *Lissajous polarization*. Exploiting the breakthrough led to calling into action the theory of Lie algebras to produce the equations of structure following the normalization.[51] Here again, the new algorithm simplifies radically the analytical calculations; at the same time, it reveals the true physical meaning of a (1-1) resonance, which the conventional treatment had buried under a heap of textbook cliches.

Handling a resonance through the classical angle, and action, variables leads to exclude critical terms from the Birkhoff normalization. The averaged system constitutes then a kind of pendulum. For so-called Ideal Resonance problems, the solution may then be pursued, in principle, by a recursive development in elliptic functions. We do not yet know how to process automatically elliptic functions and integrals in a calculus of perturbations. But, at the least, one could attempt to reduce the involvement with elliptic functions to a tolerable level. The *redressing* transformation that I propose[52] to that effect emanates from natural properties of the Lie derivative associated with the Hamiltonian of a circular pendulum. On that basis, David Richardson has developed, at the University of Cincinnati, a computer program that strips the Ideal Resonance problem of its nonessential complications to order four. Further progress in that direction promises to assist in exploring dissipative systems in which a pendulum admits a natural frequency that is slowly varying with time.[53]

The case of (1-1) resonances that are not semi-simple has also been dealt with, in principle at least[54,55]; the Birkhoff normalization for that case aims at converting the Hamiltonian into a function over the Lie algebra $su(1,1)$ relative to the group $SU(1,1)$ of unitary transformations.

How fast will experts in celestial mechanics adjust to a continuing revolution in computer technology and software invention? That is the critical question we face with the coming of LISP machines and other personal computers on one side of the scale, and of the Josephson computer on the other side. If forced to select the next challenge in celestial mechanics, I might suggest a gravitational theory of the solar system valid over the recent geological periods.

...And I was with the Sun, but no more aware
of my ascent than a man is of a thought
that comes to mind, until he finds it there.

Dante
Divine Comedy, "The Paradise," Canto 10
(Translated by John Ciardi)

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